UNIVERSITY OF COPENHAGEN
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B.Sc. Thesis

The Hurewicz Theorem

An application to the plus construction

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Abstract

In this bachelor's thesis, we shall show the absolute and relative Hurewicz Theorem for $n = 2$ using a minimum of assumptions and use the absolute Hurewicz Theorem to show the Quillen plus construction.

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COMMON SYMBOLS

$\mathbb{C}$	The complex numbers
$\mathbb{R}$	The real numbers
$\mathbb{Q}$	The rational numbers
$\mathbb{Z}$	The integers
$\mathbb{N}$	The non-negative integers $\{0, 1, 2, ...\}$
I	The closed unit interval $[0, 1]$
S^n	For $n \geqslant 0$, the unit n-sphere, $S^0 = \{-1, 1\}$
D^n	For $n \geqslant 0$, the closed unit n-disc, $D^0 = \{0\}$
∂X	The boundary of a space X
$f : X \to Y$	A continuous map f from a space X into a space Y
$f : (X, x_0) \to (Y, y_0)$	A map $f : X \to Y$ such that $f(x_0) = y_0$
$f \simeq g \operatorname{rel} x_0$	A map $f : (X, x_0) \to (Y, y_0)$ is homotopic rel x_0 to a map $g : (X, x_0) \to (Y, y_0)$
$\mathbb{1}_X$	The identity map $X \to X$
$A \hookrightarrow X$	The inclusion map from $A \subseteq X$ to X
Y^X	The space of (continuous) maps $X \to Y$
$[X, x_0, Y, y_0]$	The set of homotopy classes rel x_0 of maps $(X, x_0) \to (Y, y_0)$
$\pi_n(X, x_0)$	For $n \geqslant 1$, the n'th homotopy group of a space X with basepoint x_0
$H_n(X)$	For $n \geqslant 0$, the n'th singular homology group of a space X
$\mathbb{Z}/n$	The group of integers modulo n
F_n	The free group on n generators

$\mathbb{Z}X$	The free abelian group with basis X
$G \cong H$	A group G is isomorphic to a group H
$X \cong Y$	A space X is homeomorphic to a space Y
$X \simeq Y$	A space X is homotopy equivalent to a space Y
$c \in C$	c is an object of the category C
Obj C	The class of objects of a category C
$C(c, c')$	The class of morphisms $c \to c'$ for $c, c' \in C$
$F : C \to D$	A functor F from a category C to a category D
$\alpha : F \Rightarrow G$	A natural transformation α from a functor $F : C \to D$ to a functor $G : C \to D$
Set	The category of sets with set maps $X \to Y$ as morphims
Top	The category of topological spaces with continuous fuctions $X \to Y$ as morphisms
Top*	The category of pointed topological spaces (X, x_0) with continuous maps $f : X \to Y$ such that $f(x_0) = y_0$ as morphisms, denoted by $f : (X, x_0) \to (Y, y_0)$
Top^*_2	The category of pairs of topological spaces (X, A, x_0) such that $x_0 \in A \subseteq X$ and A is a subspace with contiuous maps $X \to Y$ as morphisms such that $f(x_0) = y_0$ and $f(A) \subseteq B$, denoted by $f : (X, A, x_0) \to (Y, B, y_0)$
Htpy	The homotopy category with homotopy sets $[X; Y]$ as morphisms
Htpy*	The homotopy category of pointed spaces iwth homotopy sets $[X, x_0; Y, y_0]$ relative to x_0 as morphisms
Grp	The category of groups with group homomorphisms $G \to H$ as morphisms
Ab	The category of abelian groups with group homomomorphisms $G \to H$ as morphisms

D^C	The category of functors $C \to D$ with natural transformations $F \Rightarrow G$ as morphisms
$\Pi_1(X)$	The fundamental groupoid of a space X with homotopy classes rel ∂I of paths from points x to y of X as morphisms
π_n	For $n \geqslant 1$, the homotopy group functor $\mathsf{Top}^* \to \mathsf{Grp}$. For $n \geqslant 2$ a functor $\mathsf{Top}^*_2 \to \mathsf{Grp}$
H_n	For $n \geqslant 0$, the singular homology group functor $\mathsf{Top} \to \mathsf{Ab}$ or $\mathsf{Top}^*_2 \to \mathsf{Ab}$
$*$	A point
$*$	Path concatenation.

CONTENTS

Abstract ◆ iii

Acknowledgements ◆ iv

Common Symbols ◆ v

Contents ◆ viii

1 Introduction ◆ 1

1.1 Basic Definitions ◆ 1

2 The Hurewicz Theorem ◆ 3

2.1 The Hurewicz homomorphisms ◆ 4

2.2 Transport along a path as group action ◆ 7

2.3 The absolute case ◆ 10

2.4 The relative case ◆ 17

3 The Quillen plus construction ◆ 19

3.1 Statement and proof ◆ 19

Bibliography ◆ 22

Chapter 1

INTRODUCTION

In this thesis, we shall examine two fundamental results of homotopy theory, first, the Hurewicz Theorem and then one of its applications, important in algebraic K-theory, the Quillen plus-construction.

The Hurewicz Theorem, which in general (that is, in terms of absolute homotopy and and homology groups), defines an isomorphism

$$h : \pi_n(X, *) \xrightarrow{\cong} H_n(X),$$

for $(n-1)$-connected spaces X for $n \geqslant 2$. We shall however restrict our proof to $n = 2$ and show which assumptions are needed in order to prove this version of the Hurewicz Theorem.

It turns out that the Quillen plus construction only needs the absolute Hurewicz theorem for $n = 2$, so our proof of the Hurewicz Theorem in Chapter 2 naturally leads us to show the statement of the Quillen plus construction in Chapter 3: That given a connected CW complex X such that $H_1(X) = 0$, there is a simply-connected space X^+ such that $H_n(X) \xrightarrow{\cong} H_n(X^+)$ for all $n \geqslant 0$ where the isomorphism is induced by the inclusion.

For the sake of completeness, we have also included a proof of the relative Hurewicz theorem for $n = 2$, which states that given a a 1-connected CW-pair, (X, A), there is an isomorphism $h^{\#} : \pi_2(X, A) \xrightarrow{\cong} H_2(X, A)$, even though we shall not treat any applications for this theorem.

We shall now give an overview of the structure of this document. Chapter 1 is the present chapter with an introduction and a bit about the mathematical conventions used. Chapter 2 contains the proof of the Hurewicz Theorems and contains four sections. Sections 2.2 and 2.4 are only relevant to the relative Hurewicz Theorem and are not used in Chapter 3 where we shall prove the Quillen plus construction using the absolute Hurewicz Theorem.

1.1 BASIC DEFINITIONS

We shall follow Riehl's definition of a category in [4, pp. 13, 17]

REMARK 1.1. The data of a functor $F : C \longrightarrow D$ will be depicted as

$$\begin{array}{ccc} C & \xrightarrow{F} & D \\ c & \longmapsto & Fc \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F(f)} \\ c' & \longmapsto & Fc' \end{array} \tag{1.1}$$

in the manner inspired by Riehl in [4].

DEFINITION 1.2 ([5, section 3.15]). Let $f : X \longrightarrow Y$ be a map. The (unreduced) mapping cylinder, M_f, is the quotient of the space $X \times I \coprod Y$ where $f(x) \sim [x, 1]$.

DEFINITION 1.3 ([5, section 3.15]). Let $f : (X, x_0) \longrightarrow (Y, y_0)$ be a map. The reduced mapping cylinder, $\tilde{M}_f$, is the quotient of the space $(X \times I)/(\{x_0\} \times I) \coprod Y$ where $f(x) \sim [x, 1]$.

THE HUREWICZ THEOREM

Chapter 2

In this chapter we shall take a look at the Hurewicz Theorem in the special case of $n = 2$, due to the absolute versions importance in the Quillen plus construction. We shall treat the Quillen plus construction in the next chapter. For completeness, we shall also treat the relative Hurewicz Theorem.

First we define the absolute and relative Hurewicz homomorphisms, then we will go though the definition of transport along a path and its interpretation as a right group action. Finally we shall prove the absolute and relative Hurewicz Theorem.

A result we shall need often is

LEMMA 2.1. *We have that* $\bigoplus_\alpha \pi_2(S^2,*) \cong \pi_2(\bigvee_\alpha S^2,*)$ *[3, p. 363], and similarlilly we have* $H_2(\bigvee_\alpha S^2) \cong \bigoplus_\alpha H_2(S^2)$ *[3, p. 126].* ■

We shall follow the proof in Theorem 8.6.4 in [6, p. 213].

Proof. We shall show the statement for π_2. By the long exact sequence of on homopty, the inclusion $\bigvee S^2_\alpha \hookrightarrow \prod_\alpha S^2$ induces an isomorphism on π_2. [3, p. 363]

By proposition 4.2 in [3, p. 343] there is an isomorphism $\varphi : \pi_2(\prod_\alpha S^2,*) \to \prod_\alpha \pi_2(S^2,*)$ given by $[f] \mapsto ([p_\alpha \circ f])_\alpha$ where p_α is the projection onto the α'th factor, and a canonical isomorphism (for finite α) $\bigoplus_\alpha \pi_2(S^2,*) \to \prod_\alpha \pi_2(S^2,*)$. Let $\Phi : \bigoplus_\alpha \pi_2(S^2,*) \to \pi_2(\bigvee_\alpha S^2,*)$ given by $([f_\alpha])_\alpha \mapsto \sum_\alpha \pi_2(i_\alpha)([f_\alpha])$. We claim that the following diagram commutes for finite α,

$$\begin{array}{ccc} \pi_2(\bigvee_\alpha S^2,*) & \xrightarrow{\cong} & \pi_2(\prod_\alpha S^2,*) \\ \Phi\uparrow & & \downarrow\varphi\ \cong \\ \bigoplus_\alpha \pi_2(S^2,*) & \xleftarrow[\cong]{} & \prod_\alpha \pi_2(S^2) \end{array} \tag{2.1}$$

Let $([f_\alpha])_\alpha \in \bigoplus_\alpha \pi_2(S^2,*)$ we have by diagram chase:

$$([f_\alpha])_\alpha \overset{\Phi}{\longmapsto} \sum_\alpha [i_\alpha \circ f_\alpha] \mapsto \sum_\alpha [i_\alpha \circ f_\alpha] \overset{\varphi}{\longmapsto} (\sum_\alpha [p_\beta \circ i_\alpha \circ f_\alpha])_\beta = ([f_\alpha])_\alpha \tag{2.2}$$

this shows that the square commutes and thus Φ is a isomorphism for finite α.

For arbitrary α: Note that the image of a continuous map of compact set, is again compact. so this means that for any $[f] \in \pi_2(\bigvee_\alpha S^2,*)$, f can only intersect a finite number of spheres, say for $\alpha \in \Lambda$ for some finite set Λ, then f is in the image $j_* : \pi_2(\bigvee_{\alpha\in\Lambda} S^2,*) \to \pi_2(\bigvee_\alpha S^2,*)$ where j is the inclusion, so by our previous considerations we have surjectivity of Φ. Let $a, b \in \bigoplus_\alpha \pi_2(S^2, *)$ and assume that $\Phi(a) = \Phi(b)$, then by definition of the direct sum, they can be represented by a finite sum, but their null-homotopies can, again by compactness, only intersect for a finite index set Λ' hence restricting Φ to Λ' we see that $a = b$, showing injectivity. ∎

2.1 THE HUREWICZ HOMOMORPHISMS

At the centre of the Hurewicz Theorem is the eponymous homomorphisms, which, under certain conditions, turns out to be isomorphisms.

There are two versions of the Hurewicz homomorphism:

$$h : \pi_n(X, x_0) \to H_n(X), \qquad n \geqslant 1 \tag{2.3}$$
$$h : \pi_n(X, A, x_0) \to H_n(X, A), \qquad n \geqslant 2. \tag{2.4}$$

The two versions of h can be disambiguated easily by looking at their domain and codomain and no confusion should arise.

We shall now construct the Hurewicz homomorphisms in the manner presented by tom Dieck [6, pp. 495–496] : We have, since there are isomorphisms

$$H_n(D^n, S^{n-1}) \cong \mathbb{Z} \cong H_n(S^n),$$

a choice of generator for each homology group $H_n(S^n)$ and $H_n(D^n, S^{n-1})$. So we choose a generator $g_n \in H_n(S^n)$ and $\tilde{g}_n \in H_n(D^n, S^{n-1})$ such that $\partial\tilde{g}_n = g_{n-1}$, and, for the quotient map $p : (D^n, S^{n-1}) \to D^n/S^{n-1}$ (note that there are a choice of homeomorphisms $D^n/S^{n-1} \cong S^n$ that we keep fixed), such that $p_*(\tilde{g}_n) = g_n$. And thus, we can fix a generator g_1, where the remaining generators are then given by induction.

We define,

Definition 2.2. The (absolute) Hurewicz map, h, is the map $\pi_n(X, x_0) \to H_n(X)$, $n \geqslant 1$, given by the assignment $[\phi] \mapsto \phi_*(g_n)$, where ϕ_* is the induced homomorphism $H_n(S^n) \to H_n(X)$. [6, p. 496]

Definition 2.3. The (relative) Hurewicz map, h, is the map $\pi_n(X, A, x_0) \to H_n(X, A)$, $n \geqslant 2$, given by the assignment $[\psi] \mapsto \psi_*(\tilde{g}_n)$, where ψ_* is the induced homomorphism $H_n(D^n, S^{n-1}) \to H_n(X, A)$. [6, p. 496]

It is clear by homotopy invariance that h is independent of the choice of representative in π_n.

Of course, the absolute and relative Hurewicz maps are more than simply maps, but crucially, *homomorphisms*, which is the next result to be shown:

Lemma 2.4 ([6, p. 496]). *The maps* h *are group homomorphisms.*

This proof is based on proposition 10.4.4 in [6, p. 254]

Proof. We show that the absolute Hurewicz map is a homomorphism. First we show that the map $\omega : [S^2, s_0, Y, y_0] \to \hom(H_2(S^2), H_2(Y))$, given by $[f] \mapsto f_*$ is a homomorphism.

Let $i_1, i_2 : S^2 \hookrightarrow S^2 \vee S^2$ and $j_1, j_2 : Y \hookrightarrow Y \vee Y$ be the inclusions to the first and second summands, respectively. Let $\Delta : H_2(S^2) \to H_2(S^2) \oplus H_2(S^2)$ be the diagonal map and let $\alpha : H_2(Y) \oplus H_2(Y) \to H_2(Y)$ be given by $(a, b) \mapsto a + b$, and $\psi^{-1} : H_2(S^2) \oplus H_2(S^2) \to H_2(S^2 \vee S^2)$ given by $(a, b) \mapsto (i_1)_*(a) + (i_2)_*(b)$ and φ are the isomorphisms mentioned in Lemma 2.1. μ and δ are the comultiplication and codiagonals, respectively.

$$\begin{array}{ccccccc} H_2(S^2) & \xrightarrow{\mu_*} & H_2(S^2 \vee S^2) & \xrightarrow{(f \vee g)_*} & H_2(Y \vee Y) & \xrightarrow{\delta_*} & H_2(Y) \\ & \searrow^{\Delta} & \downarrow{\psi}\,\cong & & \uparrow{\varphi} & \nearrow_{\alpha} & \\ & & H_2(S^2) \oplus H_2(S^2) & \xrightarrow{f_* \oplus g_*} & H_2(Y) \oplus H_2(Y) & & \end{array} \tag{2.5}$$

First, we claim that $\psi' : H_2(S^2 \vee S^2) \to H_2(S^2) \oplus H_2(S^2)$, given by

$$c \mapsto ((p_1)_*(c), (p_2)_*(c)),$$

where p_1, p_2 are the projections onto the first and second factors, respectively, equals ψ, to show this claim, note that

$$\begin{aligned} (a, b) \xmapsto{\psi^{-1}} (i_1)_*(a) + (i_2)_*(b) \xmapsto{\psi'} &\big((p_1)_*((i_1)_*(a) + (i_2)_*(b)), (p_2)_*((i_1)_*(a) + (i_2)_*(b))\big) \\ &= (a, b) \end{aligned} \tag{2.6}$$

So since we've shown that ψ^{-1} is an isomorphism in Lemma 2.1 and that ψ' is a one sided inverse to ψ^{-1} we have that $\psi = \psi'$ by uniqueness of inverses.

Note that since each projection of μ is homotopic to the identity, we have that $\psi \circ \mu_* = ((p_1 \circ \mu)_*, (p_2 \circ \mu)_*) = (\mathbb{1}_{H_2(S^2)}, \mathbb{1}_{H_2(S^2)}) = \Delta$, so It follows that the left triangle commutes.

To show that the central square commutes, let $(a, b) \in H_2(S^2) \oplus H_2(S^2)$, then we have that

$$\begin{aligned} (f \vee g)_*((i_1)_*(a) + (i_2)_*(b)) &= (f \vee g)_*((i_1)_*(a)) + (f \vee g)_*((i_2)_*(b)) \\ &= (j_1 \circ f)_*(a) + (j_2 \circ g)_*(b) \end{aligned} \tag{2.7}$$

In the other direction we have

$$(a, b) \xmapsto{f_* \oplus g_*} (f_*(a), g_*(b)) \xmapsto{\varphi} (j_1 \circ f)_*(a) + (j_2 \circ g)_*(b) \tag{2.8}$$

which shows that the central square commutes.

To show that the right triangle commutes, let $(a', b') \in H_2(Y) \otimes H_2(Y)$, then we have,

$$(a', b') \xmapsto{\varphi} (j_1)_*(a') + (j_2)_*(b') \xmapsto{\delta_*} (\delta \circ j_1)_*(a') + (\delta \circ j_2)_*(b') = a' + b', \tag{2.9}$$

which of course equals $\alpha : (a', b') \mapsto a' + b'$. showing that the map ω is homomorphism, ie. that $[f + g]_* = f_* + g_*$. Composing ω with the evaluation homomorphism gives us that the Hurewicz map is indeed a homomorphism. ∎

If the construction of the Hurewicz homomorphisms shall be considered satisfactory, we would expect the following to hold:

Lemma 2.5 ([6, p. 496]). *The homomorphisms* h *are compatible with the long exact sequences of homotopy and homology groups, in the sense that the following diagram commutes:*

$$\begin{array}{ccccc} \pi_n(X, *) & \xrightarrow{i_*} & \pi_n(X, A, *) & \xrightarrow{\partial} & \pi_{n-1}(A, *) \\ \downarrow h & & \downarrow h & & \downarrow h \\ H_n(X) & \xrightarrow{j_*} & H_n(X, A) & \xrightarrow{\partial} & H_{n-1}(A) \end{array} \tag{2.10}$$

Proof. We will show that the diagram commutes by showing that each of the constituent squares commutes separately by a diagram chase.

The left square: Let $[\phi] \in \pi_n(X, *)$, and recall that i is the inclusion.

$$[\phi] \xmapsto{i_*} [\phi] \xmapsto{h} \phi_*(\tilde{g}_n) \tag{2.11}$$

And in the other direction,

$$[\phi] \xmapsto{h} \phi_*(g_n) \xmapsto{j_*} [\phi_*(g_n)] = [(\phi \circ p)_*(\tilde{g}_n)]. \tag{2.12}$$

This shows that the first square commutes.

Now, consider a map $f : (D^n, S^{n-1}) \to (X, A)$, and let ∂f denote f restricted to S^{n-1}. Since f_* is natural we get the commutative diagram

$$\begin{array}{ccc} H_n(D^n, S^{n-1}) & \xrightarrow{\partial} & H_{n-1}(S^{n-1}) \\ \downarrow f_* & & \downarrow (\partial f)_* \\ H_n(X, A) & \xrightarrow{\partial} & H_{n-1}(A) \end{array} \tag{2.13}$$

So given a generator $\tilde{g}_n \in H_n(D^n, S^{n-1})$, we get by commutativity that

$$\partial(f_*(\tilde{g}_n)) = (\partial f)_*(\partial \tilde{g}_n). \tag{2.14}$$

Now, we are ready for another diagram chase: Let ϕ represent a homotopy class in $\pi_n(X, A)$, then we have

$$[\phi] \overset{\partial}{\longmapsto} [\partial\phi] \overset{h}{\longmapsto} (\partial\phi)_*(g_{n-1}) \tag{2.15}$$

and the other direction:

$$[\phi] \overset{h}{\longmapsto} \phi_*(\tilde{g}_n) \overset{\partial}{\longmapsto} \partial(\phi_*(\tilde{g}_n)) = (\partial\phi)_*(\partial\tilde{g}_n) = (\partial\phi)_*(g_{n-1}), \tag{2.16}$$

as wanted. ∎

2.2 TRANSPORT ALONG A PATH AS GROUP ACTION

Since the second relative homotopy groups isn't guaranteed to be abelian, the relative Hurewicz homomorphism cannot be guaranteed to be an isomorphism out of these groups. In this section we shall construct a normal subgroup $N \leqslant \pi_2(X, A, *)$ such that $\pi_2^{\#}(X, A, *) = \pi_2(X, A, *)/N$ is abelian and the Hurewicz homomorphism $\pi_2(X, A, *) \to H_2(X, A)$ induces an isomorphism $\pi_2^{\#}(X, A, *) \to H_2(X, A)$

In order to construct N, we must first introduce the notion of transporting a map $f : (I^n, \partial I^n, J^{n-1}) \to (X, A, a_0)$, $n \geqslant 2$, where $J^n = (\partial I^n \times I) \cup (I^n \times \{0\})$ along a path $\nu : I \to A$ where $\nu(0) = a_0$ and $\nu(1) = a_1$. The construction shall follow tom Dieck in sections 5.2 and 6.2 of [6]. Let $\alpha : J^{n-1} \times I \to A$ be the homotopy given by $(x, t) \mapsto \nu(t)$, so α is a homotopy between two constant paths. Note that since $J^{n-1} \hookrightarrow \partial I^n$ and $\partial I^n \hookrightarrow I^n$ are cofibrations [6, p. 127]. We can then use the Homotopy Extension Property twice, yielding the commutative diagrams below, showing that there are induced homotopies as indicated by the dashed lines.

$$\begin{array}{ccc} J^{n-1} & \xrightarrow{\alpha} & A^I \\ \big\downarrow & \overset{\beta}{\nearrow} & \big\downarrow \mathrm{ev}_0 \\ \partial I^n & \xrightarrow[f|\partial I^n]{} & A \end{array} \qquad \begin{array}{ccc} \partial I^n & \xrightarrow{\beta} & X^I \\ \big\downarrow & \overset{H}{\nearrow} & \big\downarrow \mathrm{ev}_0 \\ I^n & \xrightarrow[f]{} & X \end{array} \tag{2.17}$$

We then define:

DEFINITION 2.6. Let $\nu : I \to A$ be a path from a_0 to a_1, then the map $\nu_\# : \pi_2(X, A, a_0) \to \pi_2(X, A, a_1)$ is given by $[H_0] \mapsto [H_1]$ with H is the homotopy given above. By transport along a path ν we shall mean $\nu_\#$.

Let ν be a path in A, then we define a functor $(\cdot)_\#$ as follows:

$$\begin{array}{ccc} \Pi_1(A) & \xrightarrow{(\cdot)_\#} & \mathsf{Grp} \\ a_0 & \longmapsto & \pi_2(X,A,a_0) \\ {\scriptstyle [\nu]}\downarrow & & \downarrow{\scriptstyle \nu_\#} \\ a_1 & \longmapsto & \pi_2(X,A,a_1) \end{array} \tag{2.18}$$

Now we will show that $(\cdot)_\#$ is indeed a functor:

Proposition 2.7 ([6, p. 127]). *Transport along a path, that is, the assignment* $[H_0] \mapsto [H_1]$ *is a well-defined map* $\pi_2(X,A,a_0) \to \pi_2(X,A,a_1)$, *and a group homomorphism that depends only on the homotopy class of the path in* $\Pi_1(A)$. $(\cdot)_\#$ *is a contravariant functor from* $\Pi_1(A)$.

Proof. We shall show that $(\cdot)_\#$ is a functor. First, note that the identity $e \in \pi_1(A, a_0)$ is the homotopy class of the constant loop at a_0, $[c_{a_0}]$. So, let $[\gamma] \in \pi_2(X,A,a_0)$, so this means that the representative γ is a map $(I^2, \partial I^2, J^1) \to (X, A, a_0)$. If we let the homotopy $\alpha' : J^1 \times I \to A$ be given by $(x,t) \mapsto c_{a_0}(t)$ (we shall denote α' *the homotopy along* c_{a_0}), we get diagrams, again by the Homotopy Extension Property,

$$\begin{array}{ccc} J^1 & \xrightarrow{\alpha'} & A^I \\ \downarrow & \overset{\beta'}{\nearrow} & \downarrow{\scriptstyle ev_0} \\ \partial I^2 & \xrightarrow[\gamma|\partial I^2]{} & A \end{array} \qquad \begin{array}{ccc} \partial I^2 & \xrightarrow{\beta'} & X^I \\ \downarrow & \overset{H'}{\nearrow} & \downarrow{\scriptstyle ev_0} \\ I^2 & \xrightarrow[\gamma]{} & X \end{array} \tag{2.19}$$

This means that $(c_{a_0})_\#([\gamma]) = [H_1'] = [\gamma] \in \pi_2(X,A,a_0)$, and that for the identity element $e \in \pi_1(X,A,a_0)$ that $e_\# = \mathbb{1}_{\pi_2(X,A,a_0)}$.

Let ν be a path in A from a_0 to a_1 and let ξ be a path in A from a_1 to a_2, we then need to show that $\xi_\# \circ \nu_\# = (\nu * \xi)_\#$. Let $[f] \in \pi_2(X,A,a_0)$, then,

$$\xi_\# \circ \nu_\# : [f] \overset{\nu_\#}{\longmapsto} [G_1] \overset{\xi_\#}{\longmapsto} [H_1], \tag{2.20}$$

with G satisfying

$$\begin{array}{ccc} J^1 & \xrightarrow{\alpha} & A^I \\ \downarrow & \overset{\beta}{\nearrow} & \downarrow{\scriptstyle ev_0} \\ \partial I^2 & \xrightarrow[f|\partial I^2]{} & A \end{array} \qquad \begin{array}{ccc} \partial I^2 & \xrightarrow{\beta} & X^I \\ \downarrow & \overset{G}{\nearrow} & \downarrow{\scriptstyle ev_0} \\ I^2 & \xrightarrow[f]{} & X \end{array} \tag{2.21}$$

where α being the homotopy along ν. We have that H satisfies

$$\begin{array}{ccc} J^1 & \xrightarrow{\alpha'} & A^I \\ \downarrow & \overset{\beta'}{\nearrow} & \downarrow ev_0 \\ \partial I^2 & \xrightarrow[G_1|\partial I^2]{} & A \end{array} \qquad \begin{array}{ccc} \partial I^2 & \xrightarrow{\beta'} & X^I \\ \downarrow & \overset{H}{\nearrow} & \downarrow ev_0 \\ I^2 & \xrightarrow[G_1]{} & X \end{array} \tag{2.22}$$

where α' is the homotopy along ξ. The above means, that $\alpha * \alpha'$ is the homotopy along $\nu * \xi$. This means that $(\nu \circ \xi)_\#([f])$ satisfies

$$\begin{array}{ccc} J^1 & \xrightarrow{\alpha*\alpha'} & A^I \\ \downarrow & \overset{\beta*\beta'}{\nearrow} & \downarrow ev_0 \\ \partial I^2 & \xrightarrow[f|\partial I^2]{} & A \end{array} \qquad \begin{array}{ccc} \partial I^2 & \xrightarrow{\beta*\beta'} & X^I \\ \downarrow & \overset{G*H}{\nearrow} & \downarrow ev_0 \\ I^2 & \xrightarrow[f]{} & X \end{array} \tag{2.23}$$

So we can conclude that $(\nu * \xi)_\#([f]) = [(G * H)_1] = [H_1]$, as wanted. ∎

Remark 2.8. Note that if ν is a loop at (A, a), $\nu_\#$ is a map $\pi_2(X, A, a) \to \pi_2(X, A, a)$.

We can now define a right group action of the fundamental group $\pi_1(A, a_0)$ on the relative homotopy group $\pi_2(X, A, a_0)$ by

$$([\gamma], [\delta]) \mapsto \delta_\#(\gamma) = [\gamma] \cdot [\delta] \tag{2.24}$$

where $[\delta] \in \pi_1(A, a_0)$ and $[\gamma] \in \pi_2(X, A, a_0)$.

In order to ensure that the above assignment is indeed a (right) group action, we first need to show that $[\gamma] \cdot e = \gamma$ for all $[\gamma]$ in $\pi_2(X, A, a_0)$, and then that $([\gamma] \cdot [\delta]) \cdot [\varepsilon] = [\gamma] \cdot ([\delta * \varepsilon])$ by [6, p. 17] for $[\gamma] \in \pi_2(X, A, a_0)$ and $[\delta], [\varepsilon] \in \pi_1(A, a_0)$.

Firstly, let $e \in \pi_1(A, a_0)$ be the identity, So, let $[\gamma] \in \pi_2(X, A, a_0)$. And since $(\cdot)_\#$ is functor by Proposition 2.7, we have that $[\gamma] \cdot e = e_\#(\gamma) = \mathbb{1}_{\pi_2(X,A,a_0)}([\gamma]) = [\gamma]$, as wanted.

To show the second condition, note again that by Proposition 2.7, we have that $(\cdot)_\#$ is a functor, so this yields,

$$([\gamma] \cdot [\delta]) \cdot [\varepsilon] = \varepsilon_\#([\gamma] \cdot [\delta]) = \varepsilon_\# \circ \delta_\#([\gamma]) = (\delta * \varepsilon)_\#([\gamma]) = [\gamma] \cdot [\delta * \varepsilon], \tag{2.25}$$

as wanted.

Now we are ready the define N. Let

$$N' = \left\{ \delta(\delta \cdot z)^{-1} \mid \delta \in \pi_2(X, A, a_0), z \in \pi_1(A, a_0) \right\},$$

and let N be the normal subgroup of $\pi_2(X, A, a_0)$ generated from N'. [6, p. 496]

We have claimed that $\pi_2^\#(X, A, a_0) = \pi_2(X, A, a_0)/N$ is abelian, and in order to justify that claim, we need the following Proposition

Proposition 2.9 ([6, p. 128]). *Let $\delta, \varepsilon \in \pi_2(X, A, *)$ be given. Let $z = \partial\varepsilon \in \pi_1(A, *)$. Then $\delta \cdot z = \varepsilon^{-1}\delta\varepsilon$.* ■

So, for any $\delta, \varepsilon \in \pi_2(X, A, a_0)$, let $z = \partial\varepsilon$, we then have by Proposition 2.9 that,

$$\delta^{-1}(\delta(\delta \cdot z)^{-1})\delta = \varepsilon^{-1}\delta^{-1}\varepsilon\delta = [\varepsilon, \delta] \tag{2.26}$$

and since N is a group, we have that $[\varepsilon, \delta] \in N$, and since δ and ε are arbitrary, we have that the commutator subgroup of $\pi_2(X, A, a_0)$ is contained in N, making $\pi_2^{\#}(X, A, a_0)$ abelian, as claimed.

We shall close this section with a Lemma that shall prove useful later

Lemma 2.10 ([1, p. 476]). *Let h be the relative Hurewicz homomorphism for $n = 2$ and let N be defined as above. Then $N \subseteq \ker(h)$.*

Proof. Let $\delta \in \pi_2(X, A, *)$ and $z \in \pi_1(A, *)$, then we have,

$$h(\delta(\delta \cdot z)^{-1}) = h(\delta) - h(\delta \cdot z) = \delta_*(\tilde{g}) - (\delta \cdot z)_*(\tilde{g}) = \delta_*(\tilde{g}) - \delta_*(\tilde{g}) = 0 \tag{2.27}$$

Where the next to last equality follows since δ is freely homotopic to $\delta \cdot z$ and homotopy invariance. This shows $N' \subseteq \ker(h)$, which implies $N \subseteq \ker(h)$, as wanted. [1, p. 476] ■

2.3 The absolute case

We shall now show the absolute Hurewicz Theorem, which is the main result of this chapter, as it will be essential in proving the Quillen plus construction in Chapter 3. In this section we shall go though the various theorems, propositions and lemmas that are needed in order to reach our for this chapter which is to show the Hurewicz Theorem for simply connected CW complexes.

We shall first state two Lemmas that give exact sequences that are derived from the usual long exact sequence of homotopy and homology groups. Suppose that we have a map $f : A \to B$ and suppose furthermore that for all base points $\pi_i(CA, A, *) = 0$ for $0 < i < p$ and $\pi_i(\tilde{M}_f, A, *) = 0$ for $0 < i < q$ such that $2 < p + q - 2$, then we have the following sequences, that are used in the proof of the absolute Hurewicz theorem,

$$\pi_2(A, x_0) \to \pi_2(B, y_0) \to \pi_2(C_f, *) \to 0 \tag{2.28}$$

$$H_2(A) \to H_2(B) \to H_2(C_f) \to 0. \tag{2.29}$$

So to start this section off,

Lemma 2.11 ([5, section 3.16]). *Let* X, Y *be spaces and let* $f : (X, x_0) \to (Y, y_0)$ *be a map. For the reduced mapping cylinder* $\tilde{M}_f$ *there is a long exact sequence in homotopy*

$$\cdots \to \pi_2(X, x_0) \xrightarrow{f_*} \pi_2(Y, y_0) \to \pi_2(\tilde{M}_f, X, *) \to \pi_1(X, x_0) \to \cdots \tag{2.30}$$

■

Lemma 2.12. *Let* X, Y *be spaces and let* $f : X \to Y$ *be a map. For the mapping cylinder* M_f *there is a long exact sequence in homology*

$$\cdots \to H_2(X) \xrightarrow{f_*} H_2(Y) \to H_2(M_f, X) \to H_1(X) \to \cdots \tag{2.31}$$

Proof. The proof of this Lemma is entirely analogous to that of the previous Lemma 2.11, see [5, section 3.15]. ■

The next theorem, the Homotopy Excision Theorem by Blakers and Massey [6, p. 133], is a non-trivial result that is quite important to our task at hand, which we prove below:

Theorem 2.13 (Homotopy Excision, [6, p. 133]). *Let* $Y = Y_1 \cup Y_2$ *be a space with* Y_1, Y_2 *open and* $Y_0 = Y_1 \cap Y_2 \neq \emptyset$*. Assume that,*

$$\pi_i(Y_1, Y_0, y) = 0 \qquad \textit{for } 0 < i < p, p \geqslant 1 \tag{2.32}$$
$$\pi_i(Y_2, Y_0, y) = 0, \qquad \textit{for } 0 < i < q, q \geqslant 1 \tag{2.33}$$

for each base point $y \in Y_0$*, such that* $2 < p + q - 2$ *then the excision maps, induced by the inclusion,*

$$\iota : \pi_2(Y_2, Y_0, *) \to \pi_2(Y, Y_1, *) \tag{2.34}$$

is an isomorphism for each choice of base point $*$ *in* Y_0*.*

The proof of Excision relies on the following, theorems which we shall not prove.

Theorem 2.14 ([6, p. 144]). *Let* $(p_1, p_0) : (E_1, E_0) \to B$ *be relative a Serre fibration. Let* F_j^b *denote the fibre of* b *over* p_j*. Then the following are equivalent:*

(1) (E_1, E_0) *is* n*-connected.*

(2) (F_1^b, F_0^b) *is* n*-connected for each* b*.*

■

Theorem 2.15 ([6, p. 150]). *Suppose that the pair* (Y_1, Y_0) *is* $p-1$*-connected and* (Y_2, Y_0) *is* $q-1$*-connected for* $p, q \geqslant 0$*. Then the inclusion* $F(Y_1, Y_1, Y_0) \hookrightarrow F(Y_1, Y, Y_0)$ *is* $(p+q-3)$*-connected.* ■

And we introduce the notation ([6, p. 150]), let

$$F(Y_1, Y, Y_2) = \left\{ w \in Y^I \mid w(0) \in Y_1, w(1) \in Y_2 \right\}.$$

This proof is based on tom Dieck's proof in [6, p. 150]:

Proof of Theorem 2.13. First note that the evaluation map is always a fibration, so we get a fibration $F(Y, Y, Y_2) \to Y$ given by $\omega \mapsto \omega(0)$ which yields the pullback along the inclusion:

$$\begin{array}{ccc} F(Y,Y,Y_2) & \longrightarrow & Y \\ \uparrow & & \uparrow \\ F(Y,Y,Y_2) \times_Y Y_1 & \longrightarrow & Y_1 \end{array} \tag{2.35}$$

meaning that the map $F(Y_1, Y, Y_2) = F(Y, Y, Y_2) \times_Y Y_1 \to Y_1$ given by $\omega \mapsto \omega(0)$ is a fibration as well.

Taking the fibre of a point $* \in Y_1$ over $F(Y_1, Y_1, Y_0,) \to Y$ and $F(Y_1, Y, Y_2 \to Y) \to Y$ yields $F(\{*\}, Y_1, Y_0)$ and $F(\{*\}, Y, Y_2)$, respectively. This yields the commutative diagram

$$\begin{array}{ccc} F(\{*\}, Y_1, Y_0)) & \overset{\beta}{\hookrightarrow} & F(\{*\}, Y, Y_2) \\ \downarrow & & \downarrow \\ F(Y_1, Y_1, Y_0) & \overset{\alpha}{\hookrightarrow} & F(Y_1, Y, Y_2) \\ \downarrow & & \downarrow \\ Y_1 & \overset{=}{\longrightarrow} & Y_1 \end{array} \tag{2.36}$$

Since α is $(p+q-2)$-connected by Theorem 2.15, then β is likewise connected by Theorem 2.14.

Noting that the function space F is the same as loop space, we then have

$$\begin{array}{ccc} \pi_1(F(\{*\}, Y_1, Y_0)) & \overset{\beta_*}{\longrightarrow} & \pi_1(F(\{*\}, Y, Y_2)) \\ \downarrow \cong & & \downarrow \cong \\ \pi_2(Y_1, Y_0, *) & \longrightarrow & \pi_2(Y, Y_2, *) \end{array} \tag{2.37}$$

yielding the desired isomorphism for $2 < p+q-2$. ■

The next theorem, the Quotient Theorem, which together with a similar theorem for homology groups, establishes conditions under which $\pi_n(X, A, *) \cong \pi_n(X/A, *)$ and $H_n(X, A) \cong H_n(X/A)$. This is important, since we want to replace $\pi_2(\tilde{M}_f, X, *)$ and $H_2(M_f, X)$ with $\pi_2(\tilde{M}_f/X, *) \cong \pi_2(C_f, *)$ and $H_2(M_f/X) \cong H_2(C_f)$, respectively.

Theorem 2.16 (The Quotient Theorem, [6, p. 153]). *Assume that $A \hookrightarrow X$ is a cofibration and let $p : (X, A) \to (X/A, *)$ be the canonical quotient map. Assume further that for all $a \in A$,*

$$\pi_i(CA, A, a) = 0 \text{ for } 0 < i < m, \qquad \pi_i(X, A, a) = 0 \text{ for } 0 < i < n, \quad (2.38)$$

such that $2 < n + m - 2$, then $p_* : \pi_2(X, A, a) \to \pi_2(X/A, *)$ is an isomorphism.

Proof sketch. The idea of the proof of this theorem by tom Dieck in [6, p. 153] is to obtain an isomorphism, $\pi_2(X, A, a) \to \pi_2(X \cup CA, CA, *)$ via Homotopy Excision and then use the map induced from the projection $q : X \cup CA \to (X \cup CA)/CA$, which is a homotopy equivalence if $CA \hookrightarrow X \cup CA$ is a cofibration, thus obtaining isomorphisms

$$\pi_i(X, A, a) \to \pi_i(X \cup CA, CA, *) \to \pi_i(X \cup CA, *) \to \pi_i((X \cup CA)/CA, *) \to \pi_i(X/A), \quad (2.39)$$

with the bijection $\pi_i(X \cup CA, CA, *) \to \pi_i(X \cup CA, *)$, being a consequence of the contractibility of CA and the long exact homotopy sequence of the pair $(X \cup CA, CA)$. This yields the commutative diagram

$$\begin{array}{ccc} \pi_i(X, A, a) & \xrightarrow{\cong} & \pi_i(X \cup CA, *) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle q_*}\,{\scriptstyle \cong} \\ \pi_i(X/A, *) & \xrightarrow[\cong]{} & \pi_i((X \cup CA)/CA, *) \end{array} \quad (2.40)$$

This shows p_* to be an isomorphism (bijection/surjection).[1] ■

We shall now state the similar theorem for Homology:

Theorem 2.17 ([6, p. 254]). *Let $A \hookrightarrow X$ be a cofibration and let $p : (X, A) \to (X/A, *)$ be the projection. then $p_* : H_n(X, A) \to H_n(X/A)$ is an isomorphism.* ■

Now we are ready to prove one of the ingredients of the Hurewicz Theorem:

[1] I would like to thank Bastiaan Cnossen at Bonn for helping me understand that the commutative square showed p_* to be an isomorphism.

Lemma 2.18. *Let* $A = \bigvee_\alpha S^2$ *and* $B = \bigvee_\beta S^2$ *and let* $f : A \to B$ *be a map. Then* $\pi_2(M_f, A, *) \overset{p_*}{\cong} \pi_2(C_f, *)$. *We also have* $H_2(M_f, A) \overset{q_*}{\cong} H_2(C_f)$, *where* p *and* q *are the quotient maps.*

Proof. Firstly, note that $A \hookrightarrow M_f$ is a cofibration [6, p. 111].

Next, we check that $\pi_2(CA, A) = 0$, but since CA is contractible we have that $\pi_2(CA, A, a) \cong \pi_1(A, a) = \pi_1\left(\bigvee_\alpha S^2, *\right) \cong \bigoplus_\alpha \pi_1(S^2, *) = 0$ by lemma 2.1 on page 3.

To check that $\pi_1(CA, A, a) = 0$, note that there is a bijection of sets $\pi_1(CA, A, a) \to \pi_0(A, *)$ and since A is path connected we get that $\pi_i(CA, A, a) = 0$ for $0 < i < 3$.

Next we check that $\pi_1(M_f, A, a)$ is trivial. Note that $M_f \simeq B$, so we get an exact sequence

$$\pi_1(B, *) \to \pi_1(M_f, A, *) \to \pi_0(A, *) \to \pi_0(M_f) \tag{2.41}$$

with all the absolute homotopy groups/sets being trivial, we get $\pi_1(M_f, A, *)$ trivial as well. Then we have by theorem 2.16 that $\pi_2(M_f, A, *) \cong \pi_2(M_f/A, *)$, and the result follows since $M_f/A \cong C_f$.

We have already established that $A \hookrightarrow M_f$ is a cofibration, so we have by Theorem 2.17 that $H_2(M_f, A) \cong H_2(M_f/A) \cong H_2(C_f)$, as wanted. ∎

Remark 2.19. In the proof of Lemma 2.18, we show that $\pi_i(CA, A) = 0$ for $0 < i < 3 = p$ and $\pi_1(X, A, a) = 0$ $(q = 2)$.

This is due to the Quotient Theorem (Theorem 2.16) utilizing Homotopy Excision to show an isomorphism $\pi_2(X, A, *) \to \pi_2(X \cup CA, CA, *)$, setting $Y = X \cup CA$, $Y_1 = CA$, $Y_2 = X$ and $Y_0 = A$, which helps determine an isomorphism $\pi_2(X, A, *) \cong \pi_2(X/A, *)$ induced by the quotient map $X \to X/A$.

The next Proposition can be used to show a special case of the Hurewicz Theorem, needed in the proof:

Proposition 2.20 ([6, p. 257]). *The map* $\omega : [S^2, s_0, S^2, s_1] \to \hom(H_2(S^2), H_2(S^2))$ *given by* $[f] \mapsto f_*$ *is an isomorphism.*

This proof is based on the proof of 10.5.1 [6, p. 257]

Proof. Note, that by the proof of Lemma 2.4 the map ω is an homomorphism.

Now, we have that $[S^2, s_0, S^2, s_0] = \pi_2(S^2, s_0) \cong \mathbb{Z}$ with the identity map mapped to 1, and note that since $H_2(S^2) \cong \mathbb{Z}$ can deduce an isomorphism $\hom(H_2(S^2), H_2(S^2)) \cong \mathbb{Z}$, finishing the proof. ∎

The next Lemma is group theoretic in nature

Lemma 2.21. *Consider groups* G, H *with* G *being free on* $\{g\}$*, then the evaluation homomorphism* $\hom(G, H) \xrightarrow{ev_g} H$ *is an isomorphism.*

Proof. Note that G and H will be written additively. We have a map $ev_g : \hom(G, H) \to H$, and we shall construct an inverse $\varphi : H \to \hom(G, H)$. Let $h \in H$ we shall find a map $f \in \hom(G, H)$ such that $f(g) = h$. Since g is a generator , we can extend the map $g \mapsto h$ by linearity to G [2, p. 271] . This yields the unique homomorphism f meeting our criteria, so we set $\varphi(h) = f$. Now we show that φ is a homomorphism. Let $h, h' \in H$ and suppose that $\varphi(h) = f$ and $\varphi(h') = f'$, then $f(g) + f'(g) = h + h'$, this means that $\varphi(h + h') = f + f' = \varphi(h) + \varphi(h')$, as wanted. Next we show that ev_g and φ are mutual inverses:

$$\varphi \circ ev_g : f \xmapsto{ev_g} f(g) = h \xmapsto{\varphi} f, \tag{2.42}$$

and

$$ev_g \circ \varphi : h \xmapsto{\varphi} f \xmapsto{ev_g} f(g) = h, \tag{2.43}$$

this completes the proof. ■

Corollary 2.22 ([6, p. 497])**.** *The absolute Hurewicz homomorphism* $h : \pi_2(S^2, *) \to H_2(S^2)$ *is an isomorphism.*

Proof. Proposition 2.20, along with Lemma 2.21, states that that $h = ev_{g_2} \circ \omega$ is an isomorphism. ■

Corollary 2.23 ([6, p. 497])**.** *The absolute Hurewicz homomorphism* $h : \pi_2(\bigvee_\alpha S^2, *) \to H_2(\bigvee_\alpha S^2)$ *is an isomorphism.*

Proof. By Corollary 2.22, we have that the Hurewicz homomorphism $h : \pi_2(S^2, *) \to H_2(S^2)$ is an isomorphism, it follows that $\bigoplus_\alpha h : \bigoplus_\alpha \pi_2(S^2, *) \to \bigoplus_\alpha H_2(S^2)$ is an isomorphism as well. If the following diagram commutes, then $h : \pi_2(\bigvee_\alpha S^2, *) \to H_2(\bigvee_\alpha S^2)$ is automatically an isomorphism.

$$\begin{array}{ccc} \bigoplus_\alpha \pi_2(S^2) & \xrightarrow{\cong} & \pi_2(\bigvee_\alpha S^2) \\ {\scriptstyle \bigoplus_\alpha h} \downarrow {\scriptstyle \cong} & & \downarrow {\scriptstyle h} \\ \bigoplus_\alpha H_2(S^2) & \xrightarrow[\cong]{} & H_2(\bigvee_\alpha S^2) \end{array} \tag{2.44}$$

To check that the diagram commutes, we shall perform a diagram chase.

Let $i_\beta : S^2 \hookrightarrow \bigvee_\alpha S^2$ be the inclusion into the β'th factor and let $([f_\alpha])_\alpha \in \bigoplus_\alpha \pi_2(S^2, *)$, then the upper-right path is,

$$([f_\alpha])_\alpha \mapsto \sum_\alpha \pi_2(i_\alpha)([f_\alpha]) \overset{h}{\longmapsto} \sum_\alpha h([i_\alpha \circ f_\alpha]) = \sum_\alpha H_2(i_\alpha \circ f_\alpha)(g_2) \quad (2.45)$$

and the lower-left path,

$$([f_\alpha])_\alpha \overset{\bigoplus_\alpha h}{\longmapsto} (H_2(f_\alpha)(g_2))_\alpha \mapsto \sum_\alpha H_2(i_\alpha \circ f_\alpha)(g_2) \quad (2.46)$$

■

And the follwoing results are also needed:

Proposition 2.24 ([3, p. 353]). *Every space X has a CW approximation.* ■

Theorem 2.25 ([6, p. 213]). *Let X be a CW-complex that is simply connected, then $X \simeq Y$ where Y is a CW complex such that $Y^1 = \{*\}$.* ■

The proofs of these results where left out due to time constraints.

Now we are ready to show one of the main results of this thesis, the absolute Hurewicz Theorem:

Theorem 2.26 (The Absolute Hurewicz Theorem for $n = 2$, [6, p. 497]). *Let X be a simply connected space, then $h^\# : \pi_2^\#(X, *) \to H_2(X)$ is a isomorphism.*

This proof is based on tom Dieck's proof in [6, p. 497]

Proof. Note that since X is simply connected, it is in particular path connected, then by CW-approximation (Proposition 2.24 and Theorem 2.25), we can henceforth regard X as a CW-complex such that the 1-skeleton is $X^1 = \{*\}$. and since only 3-skeletons affect 2nd homotopy and homology group, we can regard X as a 3-dimensional CW complex. Note that since $X^1 = \{*\}$, the 2-skeleton of X is a wedge of 2-spheres and attaching 3-cells is identifying the boundary of 3-cells (ie. 2-spheres) to the bouquet of 2-spheres, which is exactly what the mapping cone C_f does, where $f : \bigvee_\alpha S^2 \to \bigvee_\beta S^2$. Deforming the tip of the mapping cone by way of a linear homotopy to the basepoint, yields a bouquet of 3-disks, and thus an homotopy equivalence $X \simeq C_f$.

Now, consider the diagram, where $A = \bigvee_\alpha S^2$ and $B = \bigvee_\beta S^2$, then by naturality and functoriality of h, we have the commuting diagram,

$$\begin{array}{ccccccc} \pi_2(A,*) & \xrightarrow{f_*} & \pi_2(B,*) & \longrightarrow & \pi_2(X,*) & \longrightarrow & 0 \\ \cong \downarrow h & & \cong \downarrow h & & \downarrow h & & \\ H_2(A) & \xrightarrow{f_*} & H_2(B) & \longrightarrow & H_2(X) & \longrightarrow & 0 \end{array} \quad (2.47)$$

And note that exactness of the rows above follows by Lemmas 2.11, 2.12 and 2.18. The maps $h : \pi_2(A, *) \to H_2(A)$ and $h : \pi_2(B) \to H_2(B)$ are isomorphisms by Corollary 2.23. Then the last map is an isomorphism by the Five-lemma, as wanted. ∎

2.4 THE RELATIVE CASE

Even though the relative Hurewicz Theorem is not used in the plus construction, due to its importance we shall shall state it here and give a proof for the case $n = 2$, where we follow tom Dieck's proof in [6, p. 500]

Note that by Lemma 2.10, we have homotopy classes in N maps to the zero element in $H_2(X, A)$, this means that from the Hurewicz homomorphism $\pi_2(X, A) \to H_2(X, A)$, we can induce a homomorphism $h^\# : \pi_2^\#(X, A) \to H_2(X, A)$ by the assignment $[[f]]_N \mapsto h([f])$.

To prove The Relative Hurewicz Theorem, we will need the following theorem: which we shall state without proof

Theorem 2.27 ([1, p. 472]). *Let $n \geqslant 2$. Let X^n be an oriented n-cell. Let $\sigma_1, \ldots, \sigma_k$ be n-cells in X^n. Consider $f : (X^n, X^n \setminus \bigcup_{i=1}^k \mathrm{Int}(\sigma_i)) \to (X, A, *)$. Then for $[f] \in \pi_n(X, A, *)^{ab}$ we have, $[f] = \sum_{i=1}^k [f|\sigma_i]$.* ∎

And the following remark 9.5.4 in [6, p. 237])

Remark 2.28. Suppose that $(X, A, *)$ is a pointed pair with A path-connected. Then $S_\bullet{}^{(X,A,*)}(X)$ is a subcomplex of $S_\bullet(X)$ such that for singular simplices σ we require that $\sigma(\Delta_k^n) \subseteq A$ and $\sigma(\Delta_k^0) = \{*\}$, then the inclusion is a chain equivalence.

We are now ready to prove the Relative Hurewicz Theorem:

Theorem 2.29 (The Relative Hurewicz Theorem for $n = 2$, [6, p. 500]). *Let X, A be a CW-pair with X and A connected. Let (X, A) be 1-connected. Then $h^\# : \pi_2^\#(X, A) \to H_2(X, A)$ is an isomorphism.*

Note that this proof is based upon tom Dieck's proof in [6, pp. 500–501] and Bredon's proof in [1, pp. 478–479].

Proof. Let $\Delta_k = [e_0, \ldots, e_k]$ be the standard k-simplex and denote its ℓ-skeleton by Δ_k^ℓ. Define $C_k^1(X, A, *)$ as the chain group spanned by chain complexes $\sigma : \Delta_k \to X$ such that $\sigma(\Delta_k^1) \subseteq A$. and $\sigma(\Delta_k^0) = \{*\}$, modulus $S_k(A)$ and let $H_2^1(X, A, *)$ be second homology group of this chain complex. Then by Remark 2.28, the inclusion is a chain equivalence and will induce an isomorphism $H_2^1(X, A, *) \cong H_2(X, A)$, since X is 1-connected.

We can without loss of generality identify representatives of homotopy classes in $\pi_2(X, A, *)$ with maps $f : (\Delta_2, \partial\Delta_2, e_0) \longrightarrow (X, A, *)$ since $D^2 \cong \Delta_2$, so we fix a homeomorphism $\varphi : D^2 \cong \Delta_2$ such that $\varphi_*(\tilde{g}_2) = [\mathbb{1}_{\Delta_2}]$ (with $\tilde{g}_n$ being as defined in section 2.1) when induced on homology.

Now consider that the Hurewicz homomorphism, $h : \pi_2^\#(X, A, *) \longrightarrow H_2(X, A)$, is the map $[f] \mapsto f_*([\mathbb{1}_{\Delta_2}])$ but since $\Delta_2^1 \subseteq \partial\Delta_2$, $f \in C_2^1(X, A, *)$, we have that $f_*([\mathbb{1}_{\Delta_2}]) = [f] \in H_2^1(X, A, *)$.

Let us now define an inverse $\psi : C_2^1(X, A, *) \longrightarrow \pi_2^\#(X, A, *)$ in this way: Let $\sigma : (\Delta_2, \partial\Delta_2, \Delta_2^0) \longrightarrow (X, A, *)$ and define

$$\psi : C_2^1(X, A, *) \ni [\sigma] \mapsto [\sigma] \in \pi_2^\#(X, A, *) \tag{2.48}$$

it is clear that if $\sigma(\Delta_2) \subseteq A$, then $[\sigma]$ is the zero homotopy class, and since $\pi_2^\#(X, A, *)$ is abelian, ψ is a well-defined homomorphism.

We need to ensure that ψ induces a well-defined homomorphism. Let σ be a basis element in $C_3^1(X, A, *)$, and let $j_* : \pi_2(X, *) \longrightarrow \pi_2(X, A, *)$ be induced by the inclusion, and note that $\partial\Delta_3 \cong S^2$, so $[g|\partial\Delta_3] \in \pi_2(X, *)$. Since $\partial_3\sigma = \sum_{i=0}^{3} \sigma \circ \delta_i^3$, where δ_i^3 are the appropriate face maps. This gives us,

$$\psi \circ \partial_3(\sigma) = \sum_{i=0}^{3} [\sigma \circ \delta_i^3] \tag{2.49}$$

$$= \pm j_*([\sigma|\partial\Delta_3]) \quad \text{by Theorem 2.27} \tag{2.50}$$

$$= 0 \quad \sigma|\partial\Delta_3 \text{ extends to } \Delta_3 \text{ [3, p. 346]}, \tag{2.51}$$

which tells us that boundaries are mapped by ψ to zero, we can induce a homomorphism on $H_2^1(X, A, *)$, it is a two sided inverse. ■

Chapter 3

THE QUILLEN PLUS CONSTRUCTION

The Quillen plus construction will enable us to take a connected CW-complex and "kill" its fundamental group, that is, modify the space X into a space X^+ that is simply connected, while at the same time preserving all homology groups.

3.1 STATEMENT AND PROOF

THEOREM 3.1 (Quillen plus construction, [3, p. 374]). *Let X be a connected CW-complex such that $H_1(X) = 0$. Then there is a simply connected CW-complex X^+ such that $X \hookrightarrow X^+$ induces isomorphisms on all homology groups.*

The proof of Theorem 3.1 draws on inspiration from [3, p. 374].

Proof. Consider a set of generators $\{[\varphi_\alpha]\}_\alpha$ of $\pi_1(X, *)$, then as maps $S^1 \to X$ – we can use these as attachment maps, forming the space $X' = X \coprod_\alpha e^2_\alpha/\sim$ by attaching 2-cells along the maps φ_α. This makes X' simply connected.

We shall now construct X^+. Consider the short exact sequence, derived from the long exact sequence for the pair $X \subseteq X'$, we have

$$0 \to H_2(X) \to H_2(X') \to H_2(X', X) \to 0 = H_1(X) \tag{3.1}$$

Note that $H_2(X', X)$ is free abelian, since

$$H_2(X', X) \cong H_2(X'/X) \cong H_2(\bigvee_\alpha S^2) \cong \bigoplus_\alpha \mathbb{Z}, \tag{3.2}$$

so the sequence above splits, yielding an isomorphism $H_2(X') \cong H_2(X) \oplus H_2(X', X)$. Note that since X' is simply connected, we have by the Absolute Hurewicz Thoerem 2.26, that there is a canonical isomorphism $\pi_2(X', *) \cong H_2(X')$, and since the summand $H_2(X', X)$ is free, we can choose a basis and represent it by a set of maps $\psi_\alpha : S^2 \to X'$ chosen so that

$$h([\psi_\alpha]) = (\psi_\alpha)_*(g_2) = (0, z_\alpha) \in H_2(X') \cong H_2(X) \oplus H_2(X', X) \tag{3.3}$$

where z_α is the α'th generator of $H_2(X', X)$. We then attach 3-cells along each ψ_α to X', obtaining X^+.

To show that our construction of X^+ has the desired properties, consider the map $\bigoplus_\alpha (\psi_\alpha)_* : \bigoplus_\alpha H_2(S^2) \to H_2(X')$ given by $(x_\alpha)_\alpha \mapsto \sum_\alpha (\psi_\alpha)_*(x_\alpha)$. Note first that since each summand of $H_2(X', X)$ is infinite cyclic, so each element of its domain (whose summands are also infinite cyclic) are mapped to an integer multiple of the generator of each summand. So we can write an element $x =$

$(x_\alpha)_\alpha \in \bigoplus_\alpha H_2(S^2)$ as $x = (n_\alpha g_2)_\alpha$ for integers n_α, but then $\bigoplus_\alpha (\psi_\alpha)_*(x) = (0, \sum_\alpha n_\alpha z_\alpha)$.

To show that $\bigoplus_\alpha (\psi_\alpha)_*$ is injective, consider elements $x = (x_\alpha)_\alpha \in \bigoplus_\alpha H_2(S^2)$ and $y = (y_\alpha)_\alpha \in \bigoplus_\alpha H_2(S^2)$ and assume that $\bigoplus_\alpha (\psi_\alpha)_*(x) = \bigoplus_\alpha (\psi_\alpha)_*(y)$. We then have

$$(0,0) = \bigoplus_\alpha (\psi_\alpha)_*(x) - \bigoplus_\alpha (\psi_\alpha)_*(y) = (0, \sum_\alpha (n_\alpha - m_\alpha) z_\alpha),$$

for integers n_α, m_α, showing that $x = y$ since the generators for each summand are distinct.

Now, let $x \in 0 \oplus H_2(X', X)$, then $x = (0, \sum_\alpha m_\alpha z_\alpha)$ for integers m_α, but then the we have that $\bigoplus_\alpha (\psi_\alpha)_*((m_\alpha g_2)_\alpha) = x$, showing that $0 \oplus H_2(X', X) \subseteq \operatorname{Im} \bigoplus_\alpha (\psi_\alpha)_*$, the reverse inclusion is clear, showing that $0 \oplus H_2(X', X) = \operatorname{Im} \bigoplus_\alpha (\psi_\alpha)_*$.

From the long exact sequence on homology of the pair (D^3, S^2) we obtain the isomorphism $H_3(D^3, S^2) \xrightarrow{\partial} H_2(S^2)$, and hence an isomorphism

$$\bigoplus_\alpha H_3(D^3, S^2) \to \bigoplus_\alpha H_2(S^2)$$

.

Note that we have $D^3/S^2 \cong S^3$. An application of the isomorphism from 2.1 and that for a cofibration $A \hookrightarrow X$, then quotient map $X \to X/A$ induces an isomorphism [6, p. 106], we also have that $X^+/X' \cong \bigvee_\alpha S^3$, thus yielding the following

$$\bigoplus_\alpha H_3(D^3, S^2) \xrightarrow{\bigoplus_\alpha p_*} \bigoplus_\alpha H_3(S^3) \xrightarrow{\bigoplus_\alpha (i_\alpha)_*} H_3(\bigvee_\alpha S^3) \xrightarrow{(q_*)^{-1}} H_3(X^+, X') \quad (3.4)$$

where i_α are inclusions and p, q are quotient maps.

The above considerations yield a commutative diagram,

$$\begin{array}{ccccccccccccc}
 & & 0 & & & & \bigoplus_\alpha H_3(D^3,S^2) & \xrightarrow{\cong} & \bigoplus_\alpha H_2(S^2) & & & & \\
 & & \uparrow & & & & \downarrow{\cong} & & \downarrow{\bigoplus_\alpha (\psi_\alpha)_*} & & & & \\
0 & \longrightarrow & H_3(X') & \longrightarrow & H_3(X^+) & \longrightarrow & H_3(X^+,X') & \xrightarrow{\partial} & H_2(X') & \longrightarrow & H_2(X^+) & \longrightarrow & 0 \\
 & & \uparrow & & & & & & & & & & \\
 & & H_3(X) & & & & & & & & & & \\
 & & \uparrow & & & & & & & & & & \\
 & & 0 & & & & & & & & & &
\end{array} \quad (3.5)$$

that gives us a short exact sequence

$$0 \to H_3(X') \to H_3(X^+) \to 0 \quad (3.6)$$

which gives isomorphisms $H_3(X) \cong H_3(X') \cong H_3(X^+)$, all of which are induced by inclusions. We also get an exact sequence

$$0 \to H_3(X', X) \xrightarrow{\partial} H_2(X') \to H_2(X^+) \to 0 \tag{3.7}$$

Note that $\operatorname{Im} \partial = 0 \oplus H_2(X', X)$ so we have that $\operatorname{Coker} \partial = H_2(X) \cong H_2(X^+)$, this isomorphism is also induced by the inclusion, as desired.

Note that since our modifications of X to obtain X^+ does not alter the homology of for $n > 3$, so we have thus proven the desired statement. ∎

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